Matemática I

Cálculo Integral (Exemplos)

Engenharia de Telecomunicações e Informática

Teresa Araújo
Departamento de Matemática

Integral Definido

Exemplo 1: $\int_0^1 e^x\,dx = \left[e^x\right]_0^1 = e^1 - e^0 = e - 1$

Exemplo 2: $\int_0^1 \frac{1}{x+1}\,dx = \left[\ln|x+1|\right]_0^1 = \ln(2) - \ln(1) = \ln(2)$

Exemplo 3:
$$\int_1^2 x^2\,dx = \left[\frac{x^3}{3}\right]_1^2 = \frac{1}{3}\left[x^3\right]_1^2 = \frac{1}{3}\left(2^3 - 1^3\right) = \frac{1}{3}(8-1) = \frac{7}{3}$$

Exemplo 4:

$$\begin{aligned}
\int_0^1 \frac{3x+2}{x^2+1}\,dx &= \int_0^1 \left(\frac{3x}{x^2+1} + \frac{2}{x^2+1}\right)\,dx \\
&= 3\int_0^1 \frac{x}{x^2+1}\,dx + 2\int_0^1 \frac{1}{x^2+1}\,dx \\
&= \frac{3}{\color{red}{2}}\int_0^1 \frac{\color{red}{2}x}{x^2+1} + 2\int_0^1 \frac{1}{x^2+1}\,dx \\
&= \left[\frac{3}{2}\ln|x^2+1| + 2\,\mathrm{arctg}(x)\right]_0^1 \\
&= \frac{3}{2}\ln(2) + 2\,\underbrace{\mathrm{arctg}(1)}_{=\pi/4} - \left(\frac{3}{2}\underbrace{\ln(1)}_{=0} + 2\,\underbrace{\mathrm{arctg}(0)}_{=0}\right) \\
&= \frac{3}{2}\ln(2) + \frac{\pi}{2}
\end{aligned}$$

Exemplo 5: Calcular $\displaystyle\int_{-2}^{1} |x+1|\,dx$

$$|x+1| = \begin{cases} x+1, & x \geq -1 \\ -x-1, & x < -1 \end{cases}$$

$$\int_{-2}^{1} |x+1|\,dx = \underbrace{\int_{-2}^{-1} (-x-1)\,dx}_{I_1} + \underbrace{\int_{-1}^{1} (x+1)\,dx}_{I_2}$$

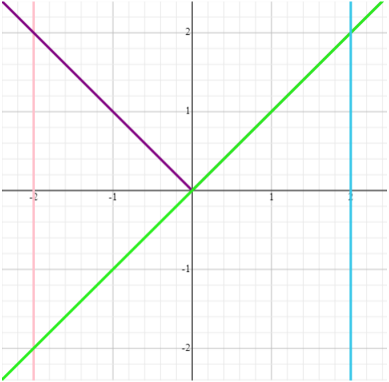
$$I_1 = \left[-\frac{x^2}{2} - x\right]_{-2}^{-1} = \left(-\frac{1}{2} - (-1)\right) - \left(-\frac{4}{2} - (-2)\right) = \frac{1}{2} - 0 = \frac{1}{2}$$

$$I_2 = \left[\frac{x^2}{2} + x\right]_{-1}^{1} = \left(\frac{1}{2} + 1\right) - \left(\frac{1}{2} + (-1)\right) = 2$$

Logo $\displaystyle\int_{-2}^{1} |x+1|\,dx = I_1 + I_2 = \frac{1}{2} + 2 = \frac{5}{2}$

Integral Definido

Exemplo 6:



$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$$\int_{-2}^{2} x\,dx = \left[\frac{x^2}{2}\right]_{-2}^{2} = 0$$

$$\int_{-2}^{2} |x|\,dx = 2\int_{0}^{2} x\,dx = 2\left[\frac{x^2}{2}\right]_{0}^{2} = 4$$

Exemplo 7: Calcule $\displaystyle\int_0^1 \frac{e^{2x}\ln(1+e^{2x})}{e^{2x}+1}\,dx$ fazendo $y = 1 + e^{2x}$

Uma vez que $y = 1 + e^{2x}$, sabemos que $e^{2x} = y - 1$, logo
$x = \dfrac{1}{2}\cdot\ln(y-1)$ pelo que $dx = \dfrac{1}{2}\dfrac{1}{y-1}\,dy$.

Além disso, temos que se $x = 0$ então $y = 1 + e^0 = 2$
e que se $x = 1$ então $y = 1 + e^2$.

Substituindo no integral vem

$$\int_0^1 \frac{e^{2x}\ln(1+e^{2x})}{e^{2x}+1}\,dx = \int_2^{1+e^2} \frac{(y-1)\ln(y)}{y}\frac{1}{2}\frac{1}{y-1}\,dy = \frac{1}{2}\int_2^{1+e^2}\frac{1}{y}\ln(y)\,dy$$
$$= \frac{1}{2}\left[\frac{1}{2}\ln^2(y)\right]_2^{1+e^2} = \frac{1}{4}\ln^2(1+e^2) - \frac{1}{4}\ln^2(2)$$

Integral Definido - Partes

Exemplo 8: Calcular $I = \displaystyle\int_{\frac{1}{2}}^{1} \operatorname{arcsen}(x)\, dx$

$$
\begin{aligned}
\int_{\frac{1}{2}}^{1} 1 \cdot \operatorname{arcsen}(x)\, dx &= \left[x \cdot \operatorname{arcsen}(x)\right]_{\frac{1}{2}}^{1} - \int_{\frac{1}{2}}^{1} x \cdot \frac{1}{\sqrt{1-x^2}}\, dx \\
&= \left[x \cdot \operatorname{arcsen}(x)\right]_{\frac{1}{2}}^{1} + \frac{1}{2} \int_{\frac{1}{2}}^{1} (-2x) \cdot (1-x^2)^{-1/2}\, dx \\
&= \left[x \cdot \operatorname{arcsen}(x) + \frac{1}{2}\frac{(1-x^2)^{1/2}}{\frac{1}{2}}\right]_{\frac{1}{2}}^{1} \\
&= \left[x \cdot \operatorname{arcsen}(x) + \sqrt{1-x^2}\right]_{\frac{1}{2}}^{1} \\
&= 1 \cdot \operatorname{arcsen}(1) + \sqrt{1-1} - \left(\frac{1}{2} \cdot \operatorname{arcsen}\left(\frac{1}{2}\right) + \sqrt{1-\frac{1}{4}}\right) \\
&= \frac{\pi}{2} - \left(\frac{1}{2} \cdot \frac{\pi}{6} + \frac{1}{2}\sqrt{3}\right) = \frac{5\pi}{12} - \frac{1}{2}\sqrt{3}
\end{aligned}
$$